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The structure of the Bern-Kosower integrand for the N-gluon amplitude

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Abstract. An ambiguity inherent in the partial integration procedure leading to the Bern-Kosower rules is fixed in a way which preserves the complete permutation symmetry in the scattering states. This leads to a canonical version of the Bern-Kosower representation for the one-loop N-photon/gluon amplitudes, and to a natural decomposition of those amplitudes into permutation symmetric gauge invariant partial amplitudes. This decomposition exhibits a simple recursive structure.

1 Introduction: The Bern-Kosower master formula

In recent years it has been found that string theory can serve as a guiding principle for the derivation of useful and non-trivial rearrangements in standard perturbative quantum field theory. While such string-related techniques have been applied to a large variety of field theory problems [1–18] the primary example is still the case of the one-loop N-photon or gluon amplitudes. A recipe for the construction of this amplitude is given by the "Bern-Kosower Rules", which originally were derived by an analysis of the infinite string tension limit of the corresponding amplitude in an appropriate string model [1–3] (see [19] for a review). A simpler derivation of the same rules was later given by Strassler in the so-called world line path integral formalism [7–10,13–17,20] (see [21] for an introductory exposition). In this approach one represents one-loop effective actions in standard quantum field theory in terms of certain first-quantized particle path integrals, and evaluates those in a way analogous to the calculation of the Polyakov path integral in string theory. The path integral relevant for the N-photon/gluon amplitude is the following [22,7]

$$\Gamma[A] = \mathrm{tr}\int_0^\infty \frac{dT}{T}\,\mathrm{e}^{-m^2T} \times \int \mathcal{D}x\,\mathcal{P}\mathrm{exp}\left[-\int_0^T d\tau\left(\frac{1}{4}\dot{x}^2 + igA_\mu\dot{x}^\mu\right)\right] \quad (1)$$

This formula expresses the one-loop effective action induced by a (complex) scalar loop with mass m for a Yang-Mills background field in terms of a quantum mechanical path integral. At fixed Schwinger proper-time T, the path integral is to be performed over the space of trajectories obeying $x(T) = x(0)$. tr denotes the global colour trace, and $\mathcal{P}$ the path-ordering of the exponential (those can be omitted in the abelian case). We use Euclidean conventions. Similar path integral representations exist for the fermion loop [23–25] and gluon loop [7,17] contributions to this amplitude.

The N-point amplitude can be extracted from this path integral by expanding the interaction term to N-th order, and then specializing to a background consisting of plane waves carrying definite polarizations ε_i and gauge algebra generators T^{a_i}. Introducing the string theoretic photon (gluon) vertex operator

$$V_i = (T^{a_i})\int_0^T d\tau_i\,\varepsilon_i\cdot\dot{x}(\tau_i)\mathrm{e}^{ik_i\cdot x(\tau_i)} \quad (2)$$

the result can be written as (for the gluon case)

$$\Gamma^{a_1\dots a_N}[k_1,\varepsilon_1;\dots;k_N,\varepsilon_N] = (-ig)^N\mathrm{tr}\int_0^\infty\frac{dT}{T}\mathrm{e}^{-m^2T}\int\mathcal{D}x(\tau)V_1V_2\cdots V_N \times\delta(\frac{\tau_N}{T})\prod_{i=1}^{N-1}\theta(\tau_i-\tau_{i+1})\mathrm{exp}\left[-\int_0^T d\tau\frac{1}{4}\dot{x}^2\right] \quad (3)$$

Here the zero on the loop has been fixed to be at the location of the N-th vertex operator. The functions $\theta(\tau_i-\tau_{i+1})$ implement the path ordering = colour ordering which one has in the non-Abelian case. This path integral is Gaussian, so that its evaluation can be done simply by "completing the square". To get an invertible kinetic term, first

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one extracts the zero mode $x_0 \equiv \frac{1}{T}\int_0^T d\tau x(\tau)$ from the path integral. The integral over x_0 is separated off, and just produces the usual energy-momentum conservation factor. The remaining path integral is then performed using the worldline Green's function

$$G_B(\tau_1,\tau_2) = \mid \tau_1 - \tau_2 \mid - \frac{(\tau_1-\tau_2)^2}{T} \tag{4}$$

Rewriting

$$\varepsilon_i \cdot \dot{x}_i \mathrm{e}^{ik_i\cdot x_i} = \mathrm{e}^{\varepsilon_i\cdot\dot{x}_i + ik_i\cdot x_i} \mid_{\mathrm{lin}(\varepsilon_i)} \tag{5}$$

one arrives at the following master formula for the scalar loop contribution to the one-loop N-gluon amplitude,

$$\begin{aligned}
&\Gamma^{a_1\dots a_N}[k_1,\varepsilon_1;\dots;k_N,\varepsilon_N] \\
&\quad = (-ig)^N \mathrm{tr}(T^{a_1}\cdots T^{a_N})(2\pi)^D \delta(\sum k_i) \\
&\qquad \times \int_0^\infty \frac{dT}{T}[4\pi T]^{-\frac{D}{2}} e^{-m^2T} \\
&\qquad \times \prod_{i=1}^N \int_0^T d\tau_i\, \delta(\frac{\tau_N}{T}) \prod_{i=1}^{N-1} \theta(\tau_i - \tau_{i+1}) \\
&\qquad \times \exp\Big\{ \sum_{i,j=1}^N \Big[\frac{1}{2} G_{Bij} k_i\cdot k_j - i\dot{G}_{Bij}\varepsilon_i\cdot k_j \\
&\qquad + \frac{1}{2}\ddot{G}_{Bij}\varepsilon_i\cdot\varepsilon_j \Big] \Big\} \mid_{\mathrm{multi-linear}}
\end{aligned} \tag{6}$$

Here it is understood that only the terms linear in all the $\varepsilon_1,\dots,\varepsilon_N$ have to be taken. Besides the Green's function G_B also its first and second derivatives appear,

$$\begin{aligned}
\dot{G}_B(\tau_1,\tau_2) &= \mathrm{sign}(\tau_1-\tau_2) - 2\frac{(\tau_1-\tau_2)}{T} \\
\ddot{G}_B(\tau_1,\tau_2) &= 2\delta(\tau_1-\tau_2) - \frac{2}{T}
\end{aligned} \tag{7}$$

Dots generally denote a derivative acting on the first variable, $\dot{G}_B(\tau_1,\tau_2) \equiv \frac{\partial}{\partial\tau_1} G_B(\tau_1,\tau_2)$, and we abbreviate G_{Bij} $\equiv G_B(\tau_i,\tau_j)$ etc.

Writing out the exponential in (6) one obtains an integrand

$$\begin{aligned}
\exp\Big\{\cdot\Big\} \mid_{\mathrm{multi-linear}} &= (-i)^N P_N(\dot{G}_{Bij},\ddot{G}_{Bij}) \\
&\quad \times \exp\Big[\frac{1}{2}\sum_{i,j=1}^N G_{Bij} k_i\cdot k_j\Big]
\end{aligned} \tag{8}$$

with a certain polynomial P_N depending on the various $\dot{G}_{Bij}, \ddot{G}_{Bij}$ and on the kinematic invariants. The resulting parameter integrals are directly related to the ones arising in a standard Feynman parameter calculation of this amplitude [26,7,27]. The exponential factor in particular will, after performance of the global T-integration, turn into the standard one-loop N-point Feynman denominator polynomial. To arrive at the Bern-Kosower rules, one now has to remove all second derivatives $\ddot{G}_{Bij}$ appearing in P_N by suitable partial integrations in the variables τ_i,

$$\begin{aligned}
&P_N(\dot{G}_{Bij},\ddot{G}_{Bij})\mathrm{e}^{\frac{1}{2}\sum G_{Bij}k_i\cdot k_j} \\
&\qquad \stackrel{\mathrm{part.int.}}{\longrightarrow} \quad Q_N(\dot{G}_{Bij})\mathrm{e}^{\frac{1}{2}\sum G_{Bij}k_i\cdot k_j}
\end{aligned} \tag{9}$$

That this is possible for any N was proven in appendix B of [2]. The new integrand is written in terms of the G_{Bij} and $\dot{G}_{Bij}$ alone, and serves as the input for the Bern-Kosower rules[1]. Those allow one to classify the various contributions to the N-photon/gluon amplitude in terms of ϕ^3-diagrams, and moreover lead to simple relations between the integrands for the scalar, spinor and gluon loop cases. A complete formulation of the rules is lengthy, and we refer the reader to [3,19]. Let us just remark that, up to global factors correcting for the differences in degrees of freedom and statistics, the integrand for the spinor loop case can be obtained from the one for the scalar loop simply by replacing every closed cycle of $\dot{G}_B$'s appearing in Q_N by its "supersymmetrization",

$$\begin{aligned}
\dot{G}_{Bi_1i_2}\dot{G}_{Bi_2i_3}\cdots\dot{G}_{Bi_ni_1} &\to \dot{G}_{Bi_1i_2}\dot{G}_{Bi_2i_3}\cdots\dot{G}_{Bi_ni_1} \\
&\quad - G_{Fi_1i_2}G_{Fi_2i_3}\cdots G_{Fi_ni_1}
\end{aligned} \tag{10}$$

where $G_{F12} = \mathrm{sign}(\tau_1-\tau_2)$ denotes the fermionic worldline Green's function. Note that an expression is considered a cycle already if it can be put into cycle form using the antisymmetry of $\dot{G}_B$ (e.g. $\dot{G}_{B12}\dot{G}_{B12} = -\dot{G}_{B12}\dot{G}_{B21}$). A similar "cycle replacement rule" holds for the gluon loop case.

Our objective in this paper is a further investigation of the partial integration procedure, and of the structure of the polynomial Q_N.

2 The $N = 4$ ambiguity and symmetric partial integration

We begin with the two-point amplitude. For $N = 2$ (8) yields

$$P_2 = \dot{G}_{B12}\varepsilon_1\cdot k_2 \dot{G}_{B21}\varepsilon_2\cdot k_1 - \ddot{G}_{B12}\varepsilon_1\cdot\varepsilon_2 \tag{11}$$

After a partial integration performed on the second term in τ_1 or τ_2 this turns into

$$Q_2 = \Big[\varepsilon_1\cdot k_2\varepsilon_2\cdot k_1 - \varepsilon_1\cdot\varepsilon_2 k_1\cdot k_2\Big]\dot{G}_{B12}\dot{G}_{B21} \tag{12}$$

We note the following two effects of this partial integration:

1. The new Feynman numerator polynomial is a function of $\dot{G}_{B12}$, and homogeneous in the external momenta k_i.
2. A transversal projector has appeared, making the gauge invariance manifest at the integrand level.

[1] We refer to the original version of these rules as given in [3, 19]. Depending on the purpose it can be preferable to proceed directly from (6) [19]

In the three-point case one finds

$$P_3=\dot{G}_{B1i}\varepsilon_1\cdot k_i\dot{G}_{B2j}\varepsilon_2\cdot k_j\dot{G}_{B3k}\varepsilon_3\cdot k_k - \Big[\ddot{G}_{B12}\varepsilon_1\cdot\varepsilon_2\dot{G}_{B3i}\varepsilon_3\cdot k_i + 2\,\text{permuted terms}\Big] \quad (13)$$

Here and in the following the dummy indices i,j,k should be summed over from 1 to N, and one has $\dot{G}_{Bii}=0$ by antisymmetry. Removing all the $\ddot{G}_{Bij}$'s by partial integrations one finds

$$\begin{aligned} Q_3=&\dot{G}_{B1i}\varepsilon_1\cdot k_i\dot{G}_{B2j}\varepsilon_2\cdot k_j\dot{G}_{B3k}\varepsilon_3\cdot k_k+\frac{1}{2}\Big\{\dot{G}_{B12}\varepsilon_1\\ &\cdot\varepsilon_2\Big[\dot{G}_{B3i}\varepsilon_3\cdot k_i\big(\dot{G}_{B1j}k_1\cdot k_j-\dot{G}_{B2j}k_2\cdot k_j\big)\\ &+\big(\dot{G}_{B31}\varepsilon_3\cdot k_1-\dot{G}_{B32}\varepsilon_3\cdot k_2\big)\dot{G}_{B3j}k_3\cdot k_j\Big]\\ &+2\text{ perm.}\Big\}=Q_3^3+Q_3^2 \end{aligned} \quad (14)$$

where

$$\begin{aligned} Q_3^3=&\dot{G}_{B12}\dot{G}_{B23}\dot{G}_{B31}Z_3(123)\\ Q_3^2=&\dot{G}_{B12}\dot{G}_{B21}Z_2(12)\dot{G}_{B3i}\varepsilon_3\cdot k_i+2\text{ perm.} \end{aligned} \quad (15)$$

We have now introduced the notation

$$\begin{aligned} Z_2(ij)\equiv&\varepsilon_i\cdot k_j\varepsilon_j\cdot k_i-\varepsilon_i\cdot\varepsilon_j k_i\cdot k_j\\ Z_n(i_1i_2\ldots i_n)\equiv&\,\mathrm{tr}\prod_{j=1}^{n}\Big[k_{i_j}\otimes\varepsilon_{i_j}-\varepsilon_{i_j}\otimes k_{i_j}\Big]\quad(n\geq 3) \end{aligned} \quad (16)$$

for the cyclically invariant Lorentz traces which appear in the result. Z_n corresponds to a $\mathrm{tr}[F^n]$ in the (abelian) effective action, and after the partial integration procedure appears multiplied by a factor of $\dot{G}_{Bi_1i_2}\dot{G}_{Bi_2i_3}\cdots\dot{G}_{Bi_ni_1}$, independently of the algorithm used [8]. The "τ-cycles" appearing in the Bern-Kosower substitution rules are thus associated to the "Lorentz cycles".

In the abelian case the three photon amplitude must vanish by Furry's theorem. To verify that this is indeed the case note that the integrand is odd under the transformation of variables $\tau_i=T-\tau_i'$, $i=1,2,3$, since

$$G_B(\tau_i,\tau_j)=G_B(\tau_i',\tau_j'),\quad \dot{G}_B(\tau_i,\tau_j)=-\dot{G}_B(\tau_i',\tau_j') \quad (17)$$

In the three-point case, Q_3 is still unique; all possible ways of performing the partial integrations lead to the same result. The same is not true any more in the four-point case, where the result of the partial integration procedure turns out to depend on the specific chain of partial integrations chosen. This ambiguity was discussed in [8], and the question asked whether some particular algorithm exists which would not single out any of the variables τ_i, and thus preserve the full permutation symmetry between the N external legs.

We will now define such an "impartial" partial integration algorithm, in the following way:

1. In every step, partially integrate away *all* $\ddot{G}_{Bij}$'s appearing in the term under inspection *simultaneously*. This is possible since different $\ddot{G}_{Bij}$'s do not share variables to being with, and this property is preserved by all partial integrations. New $\dot{G}_{Bij}$'s may be created.
2. In the first step, for every $\ddot{G}_{Bij}$ partially integrate both over τ_i and τ_j, and take the mean of the results.
3. At every following step, any $\ddot{G}_{Bij}$ appearing must have been created in the previous step. Therefore either both i and j were partially integrated over in the previous step, or just one of them. If both, the rule is to again use both variables in the actual step for partial integration, and take the mean of the results. If only one of them was used in the previous step, then the other one should be used in the actual step.

For example, the term $\ddot{G}_{B12}\ddot{G}_{B34}$ appearing in P_4 in the first step transforms as follows,

$$\begin{aligned} \ddot{G}_{B12}\ddot{G}_{B34}\to\frac{1}{4}\dot{G}_{B12}\dot{G}_{B34}\Big\{&\Big[\dot{G}_{B1i}k_1\cdot k_i-\dot{G}_{B2i}k_2\cdot k_i\Big]\\ &\times\Big[\dot{G}_{B3j}k_3\cdot k_j-\dot{G}_{B4j}k_4\cdot k_j\Big]\\ &-\ddot{G}_{B13}k_1\cdot k_3+\ddot{G}_{B14}k_1\cdot k_4\\ &+\ddot{G}_{B23}k_2\cdot k_3-\ddot{G}_{B24}k_2\cdot k_4\Big\} \end{aligned} \quad (18)$$

The terms containing a $\ddot{G}_B$ have to be further processed. Considering just the first one of them, since both variables appearing in $\ddot{G}_{B13}$ were active in the first step, both must also be used in the second one. This yields

$$\begin{aligned} &-\frac{1}{4}\dot{G}_{B12}\dot{G}_{B34}\ddot{G}_{B13}\\ &\quad\to\frac{1}{8}\dot{G}_{B12}\dot{G}_{B34}\dot{G}_{B13}\Big[\dot{G}_{B1i}k_1\cdot k_i-\dot{G}_{B3i}k_3\cdot k_i\Big]\\ &\qquad+\frac{1}{8}\dot{G}_{B13}\Big[\ddot{G}_{B12}\dot{G}_{B34}-\dot{G}_{B12}\ddot{G}_{B34}\Big] \end{aligned} \quad (19)$$

Considering again the first term in the second line, only τ_1 was active in the previous step. Therefore only τ_2 must be used now, and the third step is the final one,

$$\frac{1}{8}\dot{G}_{B13}\ddot{G}_{B12}\dot{G}_{B34}\to\frac{1}{8}\dot{G}_{B13}\dot{G}_{B12}\dot{G}_{B34}\dot{G}_{B2i}k_2\cdot k_i \quad (20)$$

This prescription treats all variables on the same footing, and therefore must lead to a permutation symmetric result. The nontrivial fact is that the process terminates after a finite number of steps, and does not become cyclic (as would be the case if, for example, one would *always* treat the indices in a $\ddot{G}_{Bij}$ symmetrically). This is not difficult to derive from the fact that, for any term in P_N, the indices appearing in the $\ddot{G}_{Bij}$'s and the first indices of the $\dot{G}_{Bij}$'s are associated to the polarization vectors, and thus must all take different values.

This algorithm transforms P_4 into

$$\begin{aligned} Q_4=&\dot{G}_{B1i}\varepsilon_1\cdot k_i\dot{G}_{B2j}\varepsilon_2\cdot k_j\dot{G}_{B3k}\varepsilon_3\cdot k_k\dot{G}_{B4l}\varepsilon_4\cdot k_l\\ &+\Big\{\frac{1}{2}\dot{G}_{B12}\varepsilon_1\cdot\varepsilon_2\Big\{\dot{G}_{B3i}\varepsilon_3\cdot k_i\dot{G}_{B4j}\varepsilon_4\cdot k_j \end{aligned}$$

$$
\begin{aligned}
&\times\Big[\dot G_{B1k}k_1\cdot k_k-\dot G_{B2k}k_2\cdot k_k\Big]\\
&+\Big[\dot G_{B3i}\varepsilon_3\cdot k_i\big(\dot G_{B41}\varepsilon_4\cdot k_1-\dot G_{B42}\varepsilon_4\cdot k_2\big)\\
&\times\dot G_{B4k}k_4\cdot k_k+\big(3\leftrightarrow 4\big)\Big]\\
&+\Big[\big(\dot G_{B31}\varepsilon_3\cdot k_1-\dot G_{B32}\varepsilon_3\cdot k_2\big)\dot G_{B43}\varepsilon_4\\
&\cdot k_3\dot G_{B4k}k_4\cdot k_k+\big(3\leftrightarrow 4\big)\Big]\Big\}+5\text{ permutations}\Big\}\\
&+\Big\{\frac{1}{4}\dot G_{B12}\dot G_{B34}\varepsilon_1\cdot\varepsilon_2\varepsilon_3\cdot\varepsilon_4\Big\{\Big[\dot G_{B1i}k_1\cdot k_i\\
&-\dot G_{B2i}k_2\cdot k_i\Big]\Big[\dot G_{B3j}k_3\cdot k_j-\dot G_{B4j}k_4\cdot k_j\Big]\\
&+\frac{1}{2}\Big[\dot G_{B13}k_1\cdot k_3-\dot G_{B23}k_2\cdot k_3-\dot G_{B14}k_1\cdot k_4\\
&+\dot G_{B24}k_2\cdot k_4\Big]\Big[\dot G_{B1i}k_1\cdot k_i+\dot G_{B2i}k_2\cdot k_i\\
&-\dot G_{B3i}k_3\cdot k_i-\dot G_{B4i}k_4\cdot k_i\Big]\Big\}+2\text{ perm.}\Big\}
\end{aligned}
\tag{21}
$$

This expression can be rewritten more compactly as follows,

$$
Q_4=Q_4^4+Q_4^3+Q_4^2-Q_4^{22}
\tag{22}
$$

where

$$
\begin{aligned}
Q_4^4=&\dot G_{B12}\dot G_{B23}\dot G_{B34}\dot G_{B41}Z_4(1234)+2\text{ permutations}\\
Q_4^3=&\dot G_{B12}\dot G_{B23}\dot G_{B31}Z_3(123)\dot G_{B4i}\varepsilon_4\cdot k_i+3\text{ perm.}\\
Q_4^2=&\dot G_{B12}\dot G_{B21}Z_2(12)\Big\{\dot G_{B3i}\varepsilon_3\cdot k_i\dot G_{B4j}\varepsilon_4\cdot k_j\\
&+\frac{1}{2}\dot G_{B34}\varepsilon_3\cdot\varepsilon_4\Big[\dot G_{B3i}k_3\cdot k_i-\dot G_{B4i}k_4\cdot k_i\Big]\Big\}\\
&+5\text{ perm.}\\
Q_4^{22}=&\dot G_{B12}\dot G_{B21}Z_2(12)\dot G_{B34}\dot G_{B43}Z_2(34)+2\text{ perm.}
\end{aligned}
\tag{23}
$$

This decomposition according to cycles is not only necessary for the application of the Bern-Kosower substitution rules, but also natural in terms of gauge invariance. The sixteen terms appearing in this decomposition are individually gauge invariant, i.e. they either vanish or turn into total derivatives if the replacement $\varepsilon_i\to k_i$ is made for any of the external legs. This is trivial for Q_4^4, Q_4^{22}, and also for Q_4^3, since if we substitute k_4 for ε_4 there (in the un-permuted term) we have a total derivative at hand,

$$
\partial_4\Big[\dot G_{B12}\dot G_{B23}\dot G_{B31}\mathrm{e}^{\frac{1}{2}G_{Bij}k_i\cdot k_j}\Big]
$$

The only not quite trivial case is a replacement of ε_3 or ε_4 in (the un-permuted term of) Q_4^2. By inspection one finds that the replacement $\varepsilon_3\to k_3$ yields the total derivative

$$
\begin{aligned}
&\partial_3\Big[\dot G_{B12}\dot G_{B21}Z_2(12)\dot G_{B4j}\varepsilon_4\cdot k_j\,\mathrm{e}^{\frac{1}{2}G_{Bij}k_i\cdot k_j}\Big]\\
&+\frac{1}{2}(\partial_3-\partial_4)\Big[\dot G_{B12}\dot G_{B21}Z_2(12)\dot G_{B34}k_3\cdot\varepsilon_4\,\mathrm{e}^{(\cdot)}\Big]
\end{aligned}
\tag{24}
$$

and analogously for ε_4. Note that the product of two-cycles Q_4^{22} appears with a minus sign in (22). The reason is that we corrected for an over-counting here; Q_4^{22} is also contained twice in Q_4^2, and separating it out from there will change the "-" to a "+".

3 Higher orders

Before proceeding to higher point amplitudes, let us further condense the notation. We thus abbreviate

$$
\begin{aligned}
\dot G_{ij}\equiv&\dot G_{Bij}\varepsilon_i\cdot k_j\\
\underline{\dot G}_{ij}\equiv&\dot G_{Bij}\varepsilon_i\cdot\varepsilon_j\\
\dot\not{G}_{ij}\equiv&\dot G_{Bij}k_i\cdot k_j\\
\dot G(i_1i_2\ldots i_n)\equiv&\dot G_{Bi_1i_2}\dot G_{Bi_2i_3}\cdots\dot G_{Bi_ni_1}Z_n(i_1i_2\ldots i_n)
\end{aligned}
\tag{25}
$$

As was mentioned before, it is known from previous work [2,3,8] that a closed "τ-cycle" $\dot G_{Bi_1i_2}\dot G_{Bi_2i_3}\cdots\dot G_{Bi_ni_1}$ after the partial integration will always appear multiplied by a complete factor of $Z_n(i_1i_2\ldots i_n)$. This motivates the last one of the abbreviations above, and also explains why the formulation of the "cycle substitution" part of the Bern-Kosower rules did not require the specification of a particular partial integration algorithm.

A given term in Q_N thus will be a product of "complete cycles" $\dot G(\cdot)$, multiplied by a remainder. Following [8] we call this remainder "tail", or "m-tail", where m denotes the number of indices not appearing in any of the cycles. For example, Q_4^2 is the product of a complete 2-cycle and a 2-tail. Only the tails depend on the choice of the partial integration algorithm. The tail generated by our specific symmetric algorithm will be denoted by $T_m(i_1\ldots i_m)$. The 1-tail is (unambiguously) given by $T_1(i)=\dot G_{ij}$ (i being fixed and j summed over).

With the above abbreviations, the result for Q_5 obtained by an application of the symmetric algorithm can be written as follows,

$$
Q_5=Q_5^5+Q_5^4+Q_5^3+Q_5^2-Q_5^{32}-Q_5^{22}
\tag{26}
$$

where

$$
\begin{aligned}
Q_5^5=&\dot G(12345)+11\text{ permutations}\\
Q_5^4=&\dot G(1234)\dot G_{5i}+14\text{ perm.}\\
Q_5^3=&\dot G(123)\Big\{\dot G_{4i}\dot G_{5j}+\frac{1}{2}\underline{\dot G}_{45}\Big[\dot\not{G}_{4i}-\dot\not{G}_{5i}\Big]\Big\}+9\text{ perm.}\\
Q_5^2=&\dot G(12)\Big\{\dot G_{3i}\dot G_{4j}\dot G_{5k}+\frac{1}{2}\underline{\dot G}_{34}\\
&\times\Big[\dot G_{5k}\Big[\dot\not{G}_{3i}-\dot\not{G}_{4i}\Big]+\dot\not{G}_{5i}\Big[\dot G_{53}-\dot G_{54}\Big]\Big]\\
&+\frac{1}{2}\underline{\dot G}_{35}\Big[\dot G_{4k}\Big[\dot\not{G}_{3i}-\dot\not{G}_{5i}\Big]+\dot\not{G}_{4i}\Big[\dot G_{43}-\dot G_{45}\Big]\Big]\\
&+\frac{1}{2}\underline{\dot G}_{45}\Big[\dot G_{3k}\Big[\dot\not{G}_{4i}-\dot\not{G}_{5i}\Big]+\dot\not{G}_{3i}\Big[\dot G_{34}-\dot G_{35}\Big]\Big]\Big\}
\end{aligned}
$$

$$
\begin{aligned}
&+9 \text{ perm.}\\
Q_5^{32}=&\dot G(123)\dot G(45)+9 \text{ perm.}\\
Q_5^{22}=&\dot G(12)\dot G(34)\dot G_{5i}+14 \text{ perm.}
\end{aligned}
\tag{27}
$$

Again we have an over-counting here; Q_5^{32} is contained once in both Q_5^3 and Q_5^2, and Q_5^{22} is contained twice in Q_5^2. And again every term appearing in this decomposition is separately gauge invariant. Let us consider only the least trivial case, which is a replacement of, say, ε_3 by k_3 in (the un-permuted term of) Q_5^2. This leads to the following total derivative,

$$
\begin{aligned}
&\partial_3\Big[\dot G(12)\dot G_{4j}\dot G_{5k}\mathrm{e}^{\frac{1}{2}G_{Bij}k_i\cdot k_j}\Big]\\
&\quad+\frac{1}{2}(\partial_3-\partial_4)\Big[\dot G(12)\underline{\dot G}_{34}\dot G_{5k}\mathrm{e}^{(\cdot)}\Big]\\
&\quad+\frac{1}{2}(\partial_3-\partial_5)\Big[\dot G(12)\underline{\dot G}_{35}\dot G_{4k}\mathrm{e}^{(\cdot)}\Big]\\
&\quad+\frac{1}{2}\partial_5\Big[\dot G(12)\underline{\dot G}_{34}\big(\dot G_{53}-\dot G_{54}\big)\mathrm{e}^{(\cdot)}\Big]\\
&\quad+\frac{1}{2}\partial_4\Big[\dot G(12)\underline{\dot G}_{35}\big(\dot G_{43}-\dot G_{45}\big)\mathrm{e}^{(\cdot)}\Big]\\
&\quad+\frac{1}{2}\partial_3\Big[\dot G(12)\underline{\dot G}_{45}\big(\dot G_{4i}-\dot G_{5i}+\dot G_{34}-\dot G_{35}\big)\mathrm{e}^{(\cdot)}\Big]
\end{aligned}
\tag{28}
$$

Comparing the 2- and 3-tails appearing in (26) with our results for $N=2,3$ we note that there is a simple relation between T_2,T_3 and Q_2,Q_3. The tail T_i can be obtained from Q_i, in its un-decomposed form, by rewriting Q_i in the tail variables, and then extending the range of all dummy indices to run over the complete set of variables $\tau_1,\ldots,\tau_5$.

It is not difficult to see that this relation generalizes to an arbitrary Q_m,T_m. Consider (the unpermuted term of) Q_N^2, which has a 2-cycle $\dot G(12)$ and a tail $T_{N-2}(3\ldots N)$. It suffices to consider those terms in Q_N having a $\varepsilon_1\cdot k_2\varepsilon_2\cdot k_1$ as their $Z_2(12)$-component. From the master formula (6) one infers that for this part of Q_N^2 the partial integration procedure can have involved only partial integrations over the tail variables $\tau_3,\ldots,\tau_N$. Thus the calculation of T_{N-2} and the lower order calculation of Q_{N-2} are identical as far as the tail indices are concerned. The presence of the cycle variables for the tail makes itself felt only through an extension of the momentum sums in the master formula, leading to the stated extension rule for dummy indices. The same type of argument shows that the structure of T_m does not depend on the number and length of the cycles it multiplies.

At this point it should be noted that every term in Q_N must have at least one cycle factor (this is a combinatorial consequence of the fact that each such term contains a total of $2N$ indices, of which only N are different). Thus the maximal tail occurring in Q_N has length $N-2$. The above connection between T_N and Q_N thus allows us to write down, *without going through the partial integration procedure again*, Q_6 as follows,

$$
\begin{aligned}
Q_6=&Q_6^6+Q_6^5+Q_6^4+Q_6^3+Q_6^2\\
&-Q_6^{42}-Q_6^{33}-Q_6^{32}-Q_6^{22}+Q_6^{222}
\end{aligned}
\tag{29}
$$

where

$$
\begin{aligned}
Q_6^6=&\dot G(123456)+\text{permutations}\quad\Big(\frac{5!}{2}=60\text{ in total}\Big)\\
Q_6^5=&\dot G(12345)T_1(6)+\text{perm.}\quad\Big(\frac{4!}{2}\binom{6}{1}=72\text{ in total}\Big)\\
Q_6^4=&\dot G(1234)T_2(56)+\text{perm.}\quad\Big(45\text{ in total}\Big)\\
Q_6^3=&\dot G(123)T_3(456)+\text{perm.}\quad\Big(20\text{ in total}\Big)\\
Q_6^2=&\dot G(12)T_4(3456)+\text{perm.}\quad\Big(15\text{ in total}\Big)\\
Q_6^{42}=&\dot G(1234)\dot G(56)+\text{perm.}\quad\Big(45\text{ in total}\Big)\\
Q_6^{33}=&\dot G(123)\dot G(456)+\text{perm.}\quad\Big(10\text{ in total}\Big)\\
Q_6^{32}=&\dot G(123)\dot G(45)T_1(6)+\text{perm.}\quad\Big(60\text{ in total}\Big)\\
Q_6^{22}=&\dot G(12)\dot G(34)T_2(56)+\text{perm.}\quad\Big(45\text{ in total}\Big)\\
Q_6^{222}=&\dot G(12)\dot G(34)\dot G(56)+\text{perm.}\quad\Big(15\text{ in total}\Big)
\end{aligned}
\tag{30}
$$

Here the only new ingredient, T_4, according to the above is related to the un-decomposed Q_4 of (21) simply by a relabelling, and an extension of the range of all dummy indices to run from 1 to 6.

Note that the integrand is not yet quite suitable for the application of the cycle substitution rules, since the tails still contain cycles. For this purpose, one should further rewrite Q_6 as

$$
\begin{aligned}
Q_6=&\hat Q_6^6+\hat Q_6^5+\hat Q_6^4+\hat Q_6^3+\hat Q_6^2\\
&+\hat Q_6^{42}+\hat Q_6^{33}+\hat Q_6^{32}+\hat Q_6^{22}+\hat Q_6^{222}
\end{aligned}
\tag{31}
$$

where the "hat" on a term means that the range of the dummy indices appearing in its tail has been restricted so as to eliminate all additional cycles. This also removes the over-counting, so that now all coefficients are unity.

It is now obvious that in this way one arrives at a canonical permutation symmetric version of the Bern-Kosower integrand for the one-loop N-photon/gluon amplitude. Moreover, this integrand naturally decomposes into gauge invariant partial amplitudes. To see the gauge invariance, note that at every step of the recursion only one new structure appears in Q_N, namely T_{N-2}. The separate gauge invariance of all terms except Q_N^2 can be inferred from the gauge invariance of lower order terms, since the total length of the cycles multiplying a given tail is clearly not relevant for this analysis. Since the complete integrand must be gauge invariant so must be Q_N^2.

In the abelian case, the final parameter integral gives the complete N-photon amplitude "in one piece",

$$
\begin{aligned}
&\Gamma[k_1,\varepsilon_1;\ldots;k_N,\varepsilon_N]\\
&\quad=(-g)^N(2\pi)^D\delta(\textstyle\sum k_i)\displaystyle\int_0^\infty\frac{dT}{T}[4\pi T]^{-\frac{D}{2}}e^{-m^2T}\\
&\qquad\times\prod_{i=1}^N\int_0^T d\tau_i\,Q_N(\dot G_{Bij})\exp\Big\{\sum_{i<j=1}^N G_{Bij}k_i\cdot k_j\Big\}
\end{aligned}
\tag{32}
$$

with no need to add permuted terms.

In the non-Abelian case the integration region is restricted by the colour ordering as in (6), so that one must explicitly sum over all non-cyclic permutations of the N gluons. Moreover, additional boundary contributions are generated in the partial integration process. Those correspond to the "tree part" of the Bern-Kosower rules, and in the effective action picture merely contribute to the covariantization of the main term [8]. To low orders it is easily verified that their contributions are given by integrals which are already known from lower-point calculations.

4 Conclusions

We expect the above construction of the Bern-Kosower integrand to be useful in future applications of the Bern-Kosower formalism beyond the on-shell five-gluon amplitude calculation of [4]. One possible application is the calculation of the QED four-photon amplitude with all four legs off-shell, which has not yet been done to the knowledge of the author. The gauge structure of the four-photon amplitude was analyzed in [28], however there the decomposition into gauge invariant partial amplitudes required an explicit solution of the Ward identities, and the result is not identical with the one reached here. The corresponding analysis for the higher point amplitudes seems not yet to have been done in field theory. As we have seen, in the present formalism a decomposition into permutation symmetric gauge invariant partial amplitudes is generated automatically if one groups the terms appearing in the final integrand according to their cycle content. Note also that the partial integration procedure makes the ultraviolet finiteness of the four-photon amplitude manifest at the integrand level, since in contrast to P_4 all terms in Q_4 have already four external momenta factored out.

The main motivation for the present investigation was, however, its potential usefulness for multiloop calculations. Equations (6) and (32) are valid off-shell[2], and thus can be used as a starting point for the construction of QED [10, 17] or QCD [16] multiloop amplitudes. In the abelian case this procedure leads to parameter integrals which combine the contributions to the photon S-matrix of all Feynman diagrams with a single scalar/spinor loop, and fixed numbers of external and internal photons [10,17]. In diagrammatic terms the partial integration procedure then effectively induces an intricate re-shuffling of terms between numbers of diagrams of various topologies, obliterating the initial connection to standard Feynman parameter integrals. A preliminary study indicates that the particular properties of the canonical integrand lead to significant calculational advantages over previous work by the author and his collaborators on QED multiloop integration as reported in [10,29]. In particular, the gauge invariant decomposition much facilitates the study of the gauge parameter dependence of the multiloop integrands. This will be discussed elsewhere.

[2] The off-shellness was not obvious in the original derivation of the Bern-Kosower master formula, since for the initial string amplitudes the requirement of conformal invariance forces the external states to be on-shell

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